

Model of a quantum particle in spacetime

Jan Naudts*

Departement Natuurkunde, Universiteit Antwerpen UIA,
Universiteitsplein 1, 2610 Antwerpen, Belgium

Maciej Kuna†

Wydział Fizyki Technicznej i Matematyki Stosowanej, Politechnika Gdańskia,
ul. Narutowicza 11/12, 80-952 Gdańsk, Poland

(Dated: v6, December 2000)

Doplicher, Fredenhagen, and Roberts [4, 6] proposed a simple model of a particle in quantum spacetime. We give a new formulation of the model and propose some small changes and additions which improve the physical interpretation. In particular, we show that the internal degrees of freedom e and m of the particle represent external forces acting on the particle. To obtain this result we follow a constructive approach. The model is formulated as a covariance system. It has projective representations in which not only the spacetime coordinates but also the conjugated momenta are two-by-two noncommuting. These momenta are of the form $P_\mu - (b/c)A_\mu$, where b is the charge of the particle. The electric and magnetic fields obtained from the vector potential A_μ coincide with the variables e and m postulated by DFR. Similarly, the spacetime position operators are of the form $Q_\mu - (al^2/\hbar c)\Omega_\mu$ where a is a generalized charge, l a fundamental length, and with vector potentials Ω_μ which are in some sense dual w.r.t. the A_μ .

I. INTRODUCTION

The DFR-model, introduced in 1994 by Doplicher, Fredenhagen, and Roberts [4, 6], describes a relativistic quantum particle with internal degrees of freedom e and m which behave under Lorentz transformations as electric resp. magnetic field vectors. It is one of the simplest models of quantum spacetime, and as such received a lot of attention in the literature. E.g. [8] adopt the basic assumption of the DFR-model that the time-position commutators $[Q_\mu, Q_\nu]$ commute with all observables.

In previous work [9, 10] we have reformulated the model as a covariance system. It is common to study a quantum system starting from a Lie group X of relevant symmetries. In a covariance system this group is supplemented with an algebra of observables which transform into each other under the actions of the symmetry group. We expect that all models of quantum mechanics and quantum field theory can be described as covariance systems. For example, standard quantum mechanics of a nonrelativistic particle can be described as a covariance system consisting of an algebra of functions of position together with the Galilei group — see [10].

In this covariance approach it is important to remember that unitary representations of a symmetry group are allowed to be projective. In particular, for the model under study, nonvanishing time-position commutators $[Q_0, Q_\alpha] \neq 0$, $\alpha = 1, 2, 3$, and nonvanishing momentum commutators $[P_\mu, P_\nu] \neq 0$ are obtained by considering projective representations of the group of shifts in spacetime and in momentum space. Indeed, let p and p' be

two shifts in momentum space, and let U denote the projective representation. Then a phase factor ξ is allowed in the composition

$$U(p)U(p') = \xi(p, p')U(p + p') \quad (1)$$

Now, write ξ into the form

$$\xi(p, p') = \exp\left(\frac{i}{2} \sum_{\mu, \nu=0}^3 p_\mu Q_{\mu, \nu} p'_\nu\right) \quad (2)$$

with Q an anti-symmetric matrix. The time-position operators Q_μ are the generators of shifts in momentum space

$$U(p) = \exp\left(-i\hbar^{-1} \sum_{\mu=0}^3 p_\mu g_{\mu, \mu} Q_\mu\right) \quad (3)$$

(the metric tensor g is diagonal with eigenvalues 1, -1 , -1 , -1). Combination of (3) and (1) implies the following commutation relations

$$[Q_\mu, Q_\nu] = -i\hbar^2 g_{\mu, \mu} Q_{\mu, \nu} g_{\nu, \nu} \quad (4)$$

In the DFR-model the r.h.s. of the latter expression is an operator which commutes with all other observables. Hence it is clear that also the phase factor $\xi(p, p')$ in (1) should be allowed to be an operator. Unitary representations with operator-valued phase factors have been studied in [9]. From a physical point of view they are acceptable if they correspond with gauge freedoms of the model, in other words, if the wavefunctions ψ and $\xi(p, p')\psi$ describe the same state of the system. This is obviously the case if $\xi(p, p')$ commutes with all observables.

Small changes of and additions to the original DFR-model are necessary to clarify the structure of the model.

*Electronic address: Jan.Naudts@ua.ac.be

†Electronic address: maciek@mifgate.mif.pg.gda.pl

In the present paper we limit ourselves to the description of a single particle. In [4, 6] also fields are considered. The technicality of the latter makes it hard to analyze the field version with the same depth as is possible for the single particle version. The main result of the present paper is the identification of the internal degrees of freedom e and m as constant external fields. It suggests that the next item to study, after the one-particle model, is not the field version of the model, but the interaction of a single particle with varying and fully quantized external fields.

An important difference with DFR is that we consider not only noncommuting spacetime coordinates but also noncommuting momentum operators. This is a deliberate choice. It is made possible by considering representations which are also projective for shifts in spacetime, the generators of which are (proportional to) the momentum operators. The consequences of making this choice will become clear further on. While considering these projective representations it turns out to be obvious to allow that the metric tensor g depends on the internal degrees of freedom e and m . We use the notation $\gamma(e, m)$ for this e, m -dependent metric tensor while g always denotes the metric tensor [1, -1, -1, -1] of Minkowski space.

Another modification to the model is the interchange of the two internal degrees of freedom e and m (corrected with a factor $e \cdot m$ to restore time reversal symmetry). This intervention is needed to allow for the interpretation of the internal degrees of freedom e and m as (analog of) electric and magnetic fields. Finally, the latter interpretation suggests the introduction of a coupling constant λ and of charges a and b .

II. THE MODEL

The internal degrees of freedom consist of two vectors e and m in \mathbf{R}^3 satisfying $|e| = |m|$ and $e \cdot m = \pm 1$. These e, m -pairs are the points of the internal configuration space Σ . It consists of two subspaces Σ_+ and Σ_- corresponding with the two possible signs of the scalar product $e \cdot m$. DFR [4, 6] give an extensive justification of this model. For our purposes it is important that under Lorentz transformations points of Σ transform into themselves. These transformations are defined as follows. Given a point e, m in Σ introduce the following antisymmetric matrix

$$\epsilon(e, m) = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & m_3 & -m_2 \\ -e_2 & -m_3 & 0 & m_1 \\ -e_3 & m_2 & -m_1 & 0 \end{pmatrix} \quad (5)$$

Let Λ be a Lorentz transformation. The transformation of $\epsilon(e, m)$ using Λ is denoted $\epsilon'(e', m')$

$$\epsilon'(e', m') = \Lambda^{-1} \epsilon(e, m) \tilde{\Lambda}^{-1} \quad (6)$$

It is again an antisymmetric matrix. It is not difficult to show that e', m' is again a point of Σ . Hence, the

Lorentz transformation Λ maps the point e, m into the point e', m' . Note that (6) differs from the conventions used in [10]. These differences are necessary because of the swap of meaning of e and m .

In the DFR-paper the variables e and m are by definition the entries of the 4-by-4 anti-symmetric matrix appearing in the commutation relations

$$[Q_\mu, Q_\nu] = i l_P^2 \mathcal{Q}_{\mu,\nu} \quad (7)$$

(l_P is Planck's length). In our notations this means that $\mathcal{Q} = \epsilon(e, m)$. Our actual result gives \mathcal{Q} proportional to $\epsilon^{-1}(e, m)$. Note that the inverse of the matrix $\epsilon(e, m)$ is given by

$$\epsilon^{-1}(e, m) = -(e \cdot m) \epsilon(m, e) \quad (8)$$

so that again the differences are explained by the interchange of e and m .

In what follows we need to integrate over Σ in a covariant way. To do so, we select a special point (e_0, m_0) in Σ satisfying $|e_0| = |m_0| = 1$. Then each Lorentz transformation Λ defines a point $(e(\Lambda), m(\Lambda))$ by the relation

$$\epsilon(e(\Lambda), m(\Lambda)) = \Lambda^{-1} \epsilon(e_0, m_0) \tilde{\Lambda}^{-1} \quad (9)$$

Then, by definition we take

$$\int_{\Sigma} de dm f(e, m) \equiv \int_{\mathcal{L}} d\Lambda f(e(\Lambda), m(\Lambda)) \quad (10)$$

where the latter is an integration over the Lorentz group \mathcal{L} . It is then obvious that the integral of $f(e, m)$ over Σ is by construction invariant under Lorentz transformations.

III. A FUNDAMENTAL LENGTH

Many authors have proposed that at very short distances the coordinates of spacetime should be discrete, or that at least Heisenberg-type of uncertainty relations should hold for time and position operators. The argument is that at the scale of Planck's length

$$l_P = \sqrt{G \hbar c^{-3}} \quad (11)$$

the quantum nature of gravitational forces is important and changes the structure of spacetime. Once that one accepts the relevance of the fundamental unit of length l_P all distances can be expressed as dimensionless numbers. In particular, one can convert inverse lengths to lengths. Using Planck's constant \hbar one can then convert momenta into lengths. In what follows we will use this idea of an absolute length l to convert shifts in position q into shifts in wavevector k by means of the relation $k = l^{-2}q$. However, this formula does not behave correctly under time reversal. In the present model we can correct for this by multiplying with the scalar product $e \cdot m$ which changes sign under time reversal, i.e. $(e \cdot m)l^{-2}q$ behaves as a wavevector (it transforms as a pseudo-vector).

Already in 1949 Born [1] suggested that, in analogy with the rest mass squared given by

$$c^{-2} \sum_{\mu,\nu=0}^3 g_{\mu,\nu} p_\mu p_\nu, \quad (12)$$

also the pseudo-distance

$$d(q, q') = \sum_{\mu,\nu=0}^3 g_{\mu,\nu} (q - q')_\mu (q - q')_\nu \quad (13)$$

could have a discrete spectrum. He proposed to introduce a new pseudo-metric, which in our notations reads

$$\sum_{\mu,\nu} g_{\mu,\nu} (q_\mu q_\nu + l^4 k_\mu k_\nu). \quad (14)$$

The group of symmetries leaving this pseudo-metric invariant is larger than the Poincaré group. By requiring covariance for this larger group extra constraints are added to the theory. See [7]. It is straightforward to see that our analysis of the DFR-model can be extended to include this larger group. However, in the present paper we restrict ourselves to the requirement of Poincaré invariance.

IV. CORRELATION FUNCTION APPROACH

The commutation relations (7) are the basis of the DFR-paper. Here, the starting point is a correlation function denoted $F(f; k, q; k', q')$, with $f(e, m)$ any function of e and m , and with k , k' , q , and q' 4-vectors (k has the meaning of a shift in the space of wavevectors, and q of a shift in space-time). Later on we construct a Hilbert space representation which is such that

$$F(f; k, q; k', q') = \langle \psi | U(k', q') \hat{f} U(k, q)^\dagger | \psi \rangle \quad (15)$$

holds. In this expression ψ is a wavefunction, $U(k, q)$ is a projective unitary representation of the additive group $\mathbf{R}^4 \times \mathbf{R}^4$ of shifts in spacetime and in wavevector space, and \hat{f} is the quantisation of the function $f(e, m)$.

The technique of constructing quantum systems starting not from commutation relations but from correlation functions has been developed recently in a mathematical paper [10]. It is a generalization of the C^* -algebraic approach which requires an algebraic structure together with correlation functions determining the state of the system. In the new approach, the C^* -algebra is replaced by a group of symmetries X acting on 'classical' functions, e.g. functions of the position of the particle, or, what we do here, functions $f(e, m)$ of the internal degrees of freedom e and m . One of the advantages of the formalism is the room it leaves for projective representations of X . This point is crucial for the present paper.

We need an explicit expression of $F(f; k, q; k', q')$ in closed form. Typically, this kind of correlation functions,

which can be expressed in closed form, describe coherent states and have a gaussian form. Our *ansatz* is

$$\begin{aligned} F(f; k, q; k', q') &= \int_{\Sigma} de dm w(e, m) f(e, m) \\ &\times \xi(k, q; k', q'; e, m) \exp \left(-\frac{1}{2\lambda} s(k, q; k', q'; e, m) \right) \end{aligned} \quad (16)$$

This expression has been obtained by elaborating the simpler versions found in [9, 10]. In this expression $w(e, m)$ is a density function, i.e. $w(e, m)$ is positive and normalized

$$\int_{\Sigma} de dm w(e, m) = 1, \quad (17)$$

$\xi(k, q; k', q'; e, m)$ is a complex phase factor, λ is a coupling constant discussed later on, and $s(k, q; k', q'; e, m)$ is a real function, bilinear in k, q and k', q' . In order to be a correlation function (16) should satisfy conditions of positivity, normalization, covariance, and continuity (see [10]). The explicit choice of $\xi(k, q; k', q'; e, m)$ and $s(k, q; k', q'; e, m)$, made below, satisfies these requirements.

The phase factor $\xi(k, q; k', q'; e, m)$ is written in the following way

$$\begin{aligned} \xi(k, q; k', q'; e, m) &= \exp \left(\frac{i}{2\lambda} (e \cdot m) u \cdot \epsilon^{-1}(e, m) u' \right) \\ \text{where } u &= lk + \lambda l^{-1} \eta(e, m) q \\ \text{and } u' &= lk' + \lambda l^{-1} \eta(e, m) q' \end{aligned} \quad (18)$$

It involves the 4-by-4 matrix η given by

$$\eta(e, m) = (e \cdot m) \epsilon(e, m) \gamma^{-1}(e, m) \quad (19)$$

Its function is to transform positions into wavevectors. As discussed before, the factor $(e \cdot m)$ is necessary because wavevectors are pseudovectors changing sign under time reversal. The choice of η has to be made in such a way that

$$\eta(e', m') = \Lambda^{-1} \eta(e, m) \Lambda \quad (20)$$

holds for any proper Lorentz transformation Λ , when e', m' are related to e, m via (6). This condition is satisfied if the matrix $\gamma(e, m)$ transforms like ϵ , i.e.

$$\gamma(e', m') = \Lambda^{-1} \gamma(e, m) \tilde{\Lambda}^{-1} \quad (21)$$

should hold. Note that $\gamma(e, m) = g$ satisfies the latter condition. Throughout this paper one can substitute $\gamma(e, m)$ by g . Note that we assume in the sequel that $\gamma(e, m)$ is a symmetric matrix.

The function $s(k, q; k', q'; e, m)$ is given by

$$\begin{aligned} s(k, q; k', q'; e, m) &= u \cdot T(e, m) u \\ \text{where } u &= l(k - k') + \lambda l^{-1} \eta(e, m) (q - q') \end{aligned} \quad (22)$$

It involves a symmetric 4-by-4 matrix $T_{\mu,\nu}(e, m)$ whose elements may depend on e and m . At first sight the

expression $\exp(-(1/2\lambda)s(k, q; k', q'; e, m))$ does not look Lorentz-covariant. It is indeed necessary to make a special 'covariant' choice of the matrix $T_{\mu,\nu}(e, m)$. The requirement of covariance turns out to be that it should transform in the same way as $\epsilon(e, m)$. This means that, if the Lorentz transformation Λ transforms $\epsilon(e, m)$ into $\epsilon(e', m')$ then also

$$T(e', m') = \Lambda^{-1} T(e, m) \tilde{\Lambda}^{-1} \quad (23)$$

holds. Assume e.g. that $T(e, m) = (1/2)\mathbf{I}$ (half the identity matrix) whenever the length of e and m is equal to 1. Next define $T(e, m)$ for arbitrary e and m by $T(e, m) = (1/2)\Lambda^{-1}\tilde{\Lambda}^{-1}$ where Λ is any Lorentz boost for which $\epsilon(e, m) = \Lambda^{-1}\epsilon(e_0, m_0)\tilde{\Lambda}^{-1}$ with e_0 and m_0 vectors of length 1.

V. HILBERT SPACE REPRESENTATION

The correlation function (16) can be used to define a scalar product for wavefunctions of the form $\psi(k, q, e, m)$ by the formula

$$\begin{aligned} \langle \psi | \phi \rangle &= \int_{\Sigma} de dm \int_{\mathbf{R}^4} dk \int_{\mathbf{R}^4} dq \int_{\mathbf{R}^4} dk' \int_{\mathbf{R}^4} dq' \\ &\quad \times \phi(k, q, e, m) \bar{\psi}(k', q', e, m) \xi(k, q; k', q'; e, m) \\ &\quad \times \exp\left(-\frac{1}{2\lambda}s(k, q; k', q'; e, m)\right) \end{aligned} \quad (24)$$

This scalar product defines the Hilbert space of wavefunctions. We cannot use the more common representation involving square integrable wavefunctions. Therefore one should be careful with the meaning of $|\psi(k, q, e, m)|^2$ being the probability density of shifts in k - and q -space and of internal degrees of freedom e and m .

In this Hilbert space exists a unitary representation of shifts in k - and q -space. It is given by

$$\begin{aligned} &U(k, q)\psi(k', q', e, m) \\ &= \psi(k + k', q + q', e, m)\xi(k', q'; k, q; e, m) \end{aligned} \quad (25)$$

The representation is projective. Indeed, one verifies immediately, using (25), that

$$U(k, q)U(k', q') = \hat{\xi}(k, q; k', q')U(k + k', q + q') \quad (26)$$

We use a $\hat{\cdot}$ to denote multiplication operators. So, if f is a function of k, q, e, m then \hat{f} is the operator which multiplies $\psi(k, q, e, m)$ with $f(k, q, e, m)$. In particular, $\hat{\xi}(k', q'; k'', q'')$ is the operator which multiplies $\psi(k, q, e, m)$ with $\xi(k', q'; k'', q''; e, m)$.

The correlation functions $F(f; k, q; k', q')$ follow from equations (15, 24) if the wavefunction ψ is taken as

$$\psi(k, q, e, m) = \delta(k)\delta(q)\sqrt{w(e, m)} \quad (27)$$

with $\delta(k)$ and $\delta(q)$ Dirac's delta function. Remember that the wavefunctions are *not* necessarily square integrable functions so that the choice (27) is acceptable.

On the other hand, the interpretation of $|\psi(k, q, e, m)|^2$ as a probability density of finding the quantum particle in the state k, q, e, m is *not* correct. This will be clear from the explicit expression for position and momentum operators as given in the next section.

VI. POSITION AND MOMENTUM OPERATORS

The position and momentum operators Q_μ and $P_\mu = \hbar K_\mu$ are by definition the generators of the group of shifts in wavevector space resp. spacetime. Let us fix conventions in such a way that

$$U(k, q) = \exp(-ik \cdot \hat{\gamma}^{-1}Q + i\hat{\gamma}^{-1}q \cdot K) \quad (28)$$

holds. A quick calculation using (26) gives then a result which can be written as

$$\begin{aligned} Q_\mu &= \sum_{\nu=0}^3 \hat{\gamma}_{\mu,\nu} i \frac{\partial}{\partial k_\nu} + \frac{1}{2} \hat{q}_\mu - \frac{al^2}{\hbar c} \hat{\Omega}_\mu \\ K_\mu &= - \sum_{\nu=0}^3 \hat{\gamma}_{\mu,\nu} i \frac{\partial}{\partial q_\nu} + \frac{1}{2} \hat{k}_\mu - \frac{b}{\hbar c} \sigma_3 \hat{A}_\mu \end{aligned} \quad (29)$$

with σ_3 the operator which multiplies the wavefunction $\psi(k, q, e, m)$ with $e \cdot m$, with a and b 'charges' of the particle, with Ω_μ given by

$$\Omega_\mu(k, e, m) = -\frac{\hbar c}{2\lambda a} \sum_{\nu=0}^3 \eta_{\mu,\nu}^{-1}(e, m) k_\nu \quad (30)$$

and with A_μ given by

$$A_\mu(q, e, m) = -(e \cdot m) \frac{\lambda \hbar c}{2bl^2} \sum_{\nu=0}^3 \eta_{\mu,\nu}(e, m) q_\nu \quad (31)$$

The quantities $A_\mu(q, e, m)$ form a vector potential. They satisfy the rather unusual condition

$$\begin{aligned} &\sum_{\mu,\nu} \gamma_{\mu,\nu}^{-1}(e, m) q_\mu A_\nu(q, e, m) \\ &= -\frac{\lambda}{b} \frac{\hbar c}{2l^2} \gamma^{-1}(e, m) q \cdot \epsilon(e, m) \gamma^{-1}(e, m) q \\ &= 0 \end{aligned} \quad (32)$$

Introduce the notations

$$E_\alpha = \sum_{\nu} \gamma_{0,\nu}(e, m) \frac{\partial A_\alpha}{\partial q_\nu} - \sum_{\nu} \gamma_{\alpha,\nu}(e, m) \frac{\partial A_0}{\partial q_\nu}, \quad \alpha = 1, 2, 3 \quad (33)$$

and

$$B_\alpha = - \sum_{\nu=0}^3 \sum_{\beta,\zeta=1}^3 \bar{\varepsilon}_{\alpha,\beta,\zeta} \gamma_{\zeta,\nu}(e, m) \frac{\partial A_\beta}{\partial q_\nu}, \quad \alpha = 1, 2, 3 \quad (34)$$

with $\bar{\varepsilon}_{\alpha,\beta,\zeta}$ the fundamental antisymmetric tensor of dimension 3. One calculates

$$\begin{aligned} E_\alpha &= -(e \cdot m) \frac{\lambda}{b} \frac{\hbar c}{2l^2} \left(\sum_\nu \gamma_{0,\nu} \eta_{\alpha,\nu} - \sum_\nu \gamma_{\alpha,\nu} \eta_{0,\nu} \right) \\ &= \frac{\lambda \hbar c}{b l^2} e_\alpha \end{aligned} \quad (35)$$

and

$$\begin{aligned} B_\alpha &= (e \cdot m) \frac{\lambda \hbar c}{b 2l^2} \sum_{\nu=0}^3 \sum_{\beta,\zeta} \bar{\varepsilon}_{\alpha,\beta,\zeta} \gamma_{\zeta,\nu} \eta_{\beta,\nu} \\ &= \frac{\lambda \hbar c}{b l^2} m_\alpha \end{aligned} \quad (36)$$

Assume now that λ is the fine structure constant of electromagnetism and that b is the charge of the proton. They are related by

$$b^2 = \lambda \hbar c \quad (37)$$

Then the equations (35, 36) become

$$\begin{aligned} E_\alpha &= \frac{b}{l^2} e_\alpha \\ B_\alpha &= \frac{b}{l^2} m_\alpha \end{aligned} \quad (38)$$

Note that b/l^2 is the strength of the electric field of the proton at distance l . One concludes that e_α and m_α can be interpreted as a magnetic resp. electric field, measured in absolute units which relate to the elementary charge b and the intrinsic length l .

In analogy with (32), the $\Omega_\mu(k, e, m)$ satisfy the condition

$$\begin{aligned} &\sum_\mu \gamma_{\mu,\nu}^{-1}(e, m) k_\mu \Omega_\nu(k, e, m) \\ &= -\frac{\hbar c}{2\lambda a} \sum_{\mu,\nu,\zeta} \gamma_{\mu,\nu}^{-1}(e, m) k_\mu \eta^{-1}(e, m)_{\nu,\zeta} k_\zeta \\ &= -\frac{\hbar c}{2\lambda a} (e \cdot m) k \cdot \epsilon^{-1}(e, m) k \\ &= 0 \end{aligned} \quad (39)$$

The fields E_α and B_α can be obtained from $\Omega_\mu(k, e, m)$ by

$$\begin{aligned} E_\alpha &= \frac{\lambda a}{l^2 b} \sum_{\beta,\zeta=1}^3 \sum_{\nu=0}^3 \bar{\varepsilon}_{\alpha,\beta,\zeta} \gamma_{\zeta,\nu}^{-1}(e, m) \frac{\partial \Omega_\nu}{\partial k_\beta} \\ B_\alpha &= \frac{\lambda a}{l^2 b} \left[\sum_{\nu=0}^3 \gamma_{0,\nu}^{-1}(e, m) \frac{\partial \Omega_\nu}{\partial k_\alpha} - \sum_{\nu=0}^3 \gamma_{\alpha,\nu}^{-1}(e, m) \frac{\partial \Omega_\nu}{\partial k_0} \right] \end{aligned} \quad (40)$$

($\alpha = 1, 2, 3$). The symmetry between these relations and (33, 34) can be understood because the matrices $\frac{\partial \Omega_\mu}{\partial k_\nu}$ and $\frac{\partial A_\mu}{\partial q_\nu}$ are each others inverses (up to a constant factor). Indeed, one has

$$\sum_{\nu=0}^3 \frac{\partial A_\mu}{\partial q_\nu} \frac{\partial \Omega_\nu}{\partial k_\sigma} = (e \cdot m) \frac{\hbar^2 c^2}{4ab l^2} \delta_{\mu,\sigma} \quad (41)$$

with $\delta_{\mu,\sigma}$ Kronecker's delta.

VII. COMMUTATION RELATIONS

From (29) one obtains the following commutation relations

$$\begin{aligned} [Q_\mu, Q_\nu] &= -\frac{al^2}{\hbar c} \sum_{\sigma=0}^3 \left[i \frac{\partial}{\partial k_\sigma}, \hat{\gamma}_{\mu,\sigma} \hat{\Omega}_\nu - \hat{\gamma}_{\nu,\sigma} \hat{\Omega}_\mu \right] \\ &= i \frac{l^2}{2\lambda} \sum_{\sigma=0}^3 (\hat{\gamma}_{\mu,\sigma} \hat{\eta}_{\nu,\sigma}^{-1} - \hat{\gamma}_{\nu,\sigma} \hat{\eta}_{\mu,\sigma}^{-1}) \\ &= -i \frac{l^2}{\lambda} \sigma_3 (\hat{\gamma} \hat{\epsilon}^{-1} \hat{\gamma})_{\mu,\nu} \end{aligned} \quad (42)$$

and

$$\begin{aligned} [K_\mu, K_\nu] &= \frac{\lambda}{\hbar c} \sigma_3 \sum_{\sigma=0}^3 \left[i \frac{\partial}{\partial q_\sigma}, \hat{\gamma}_{\mu,\sigma} \hat{A}_\nu - \hat{\gamma}_{\nu,\sigma} \hat{A}_\mu \right] \\ &= -i \frac{\lambda}{2l^2} \sum_{\sigma=0}^3 (\hat{\gamma}_{\mu,\sigma} \hat{\eta}_{\nu,\sigma} - \hat{\gamma}_{\nu,\sigma} \hat{\eta}_{\mu,\sigma}) \\ &= i \frac{\lambda}{l^2} \sigma_3 \hat{\epsilon}_{\mu,\nu} \end{aligned} \quad (43)$$

and

$$[K_\mu, Q_\nu] = -i \hat{\gamma}_{\mu,\nu} \quad (44)$$

As explained before, the main difference between (42) and (7) comes from the interchange of e and m . Further differences are the appearance in (42) of the inverse of the coupling constant λ and of the generalized metric tensor $\gamma(e, m)$. If $\gamma(e, m) \equiv g$ then the only effect is a change of sign for the commutator between the time operator and the position operators. The appearance of factors $\gamma(e, m)$ in (42) and (44) is a consequence of including $\gamma^{-1}(e, m)$ in the definition (25) of the generators K_μ and Q_μ .

Many authors, e.g. [3, 5], have studied noncanonical commutation relations comparable with (42, 43, 44) – see e.g. the references cited in [4, 6]. A review of these works is out of scope of the present paper.

Note that, if one takes $\hat{\Omega}_\mu$ and \hat{A}_μ equal to zero in (29) then one obtains a representation describing a particle of mass zero in the off-shell formalism of relativistic quantum mechanics. The noncanonical commutation relations, which we have here, are a consequence of the presence in (29) of terms containing $\hat{\Omega}_\mu$ resp. \hat{A}_μ . Now, the procedure of replacing momenta P_μ by new momenta $P_\mu - (b/c)A_\mu$ is well-known from electrodynamics. Note that the components of the new momenta $P_\mu - (b/c)A_\mu$ do not necessarily commute between themselves (this fact is well known, and was used e.g. in [3] as an argument to introduce noncommuting position operators). Hence noncommuting momenta are quite common in quantum electrodynamics. In the present model there is not only a substitution of P_μ by $P_\mu - \sigma_3(b/c)\hat{A}_\mu$ but also a substitution of Q_μ by $Q_\mu - (al^2/\hbar c)\hat{\Omega}_\mu$. The latter is responsible for the nonvanishing time-position commutators.

VIII. POINCARÉ INVARIANCE

Shifts of the particle in spacetime are described by the unitary operators $U(0, q)$. Indeed, one verifies that

$$U(0, q)Q_\mu U(0, q)^\dagger = Q_\mu + q_\mu \quad (45)$$

On the other hand is

$$U(0, q)K_\mu U(0, q)^\dagger = K_\mu + \frac{b}{l^2} \sum_\nu \hat{\eta}_{\mu,\nu} q_\nu \quad (46)$$

Clearly, the operators K_μ are not conserved under shifts in spacetime. This is understandable because the particle moves in external fields.

Similarly, shifts in the space of wavevectors are described by the unitary operators $U(k, 0)$. Indeed, one has

$$U(k, 0)K_\mu U(k, 0)^\dagger = K_\mu + k_\mu \quad (47)$$

Next we define a unitary representation R of the proper Lorentz group. The *ansatz* is

$$R(\Lambda)\psi(k, q, e, m) = \psi(\Lambda^{-1}k, \Lambda^{-1}q, e', m') \quad (48)$$

with e', m' related to e, m by (6). The conjugate operator $R(\Lambda)^\dagger$ is given by

$$R(\Lambda)^\dagger\psi(k, q, e', m') = \psi(\Lambda k, \Lambda q, e, m) \quad (49)$$

It is now straightforward to verify that $R(\Lambda)$ is a unitary representation of the proper Lorentz group.

We cannot use (48) for the whole of the Lorentz group because time reversal must be implemented as an anti-unitary operator [2] because under time reversal q_μ goes into $-g_{\mu,\mu}q_\mu$ while k_μ goes into $g_{\mu,\mu}k_\mu$. The operator Θ given by

$$\Theta\psi(k, q, e, m) = \overline{\psi(gk, -gq, -e, m)} \quad (50)$$

satisfies all requirements. It obviously satisfies $\Theta^2 = \mathbf{1}$. One verifies that

$$(\Theta\phi, \psi) = (\Theta\psi, \phi) \quad (51)$$

Finally, the parity operator P is defined as an isometry between Hilbert spaces by

$$P\psi(k, q, e, m) = \psi(gk, gq, -e, m) \quad (52)$$

The parity-inverted scalar product is given by

$$\begin{aligned} \langle \psi | \phi \rangle' &= \int_{\Sigma} de dm \int_{\mathbf{R}^4} dk \int_{\mathbf{R}^4} dq \int_{\mathbf{R}^4} dk' \int_{\mathbf{R}^4} dq' \\ &\times \phi(k, q, e, m) \overline{\psi(k', q', e, m)} \\ &\times \xi(gk, gq; gk', gq'; -e, m) \\ &\times \exp\left(-\frac{1}{2\lambda} s(gk, gq; gk', gq'; -e, m)\right) \end{aligned} \quad (53)$$

It satisfies

$$\langle P\psi | P\phi \rangle' = \langle \psi | \phi \rangle \quad (54)$$

IX. INVARIANTS

The position and wavevector operators Q_μ and K_μ transform as expected under proper Lorentz transformations. From (48, 49) and the definitions (29) one obtains

$$\begin{aligned} R(\Lambda)Q_\mu R(\Lambda)^\dagger &= \sum_\nu \Lambda_{\mu,\nu}^{-1} Q_\nu \\ R(\Lambda)K_\mu R(\Lambda)^\dagger &= \sum_\nu \Lambda_{\mu,\nu}^{-1} K_\nu \end{aligned} \quad (55)$$

Note that also

$$R(\Lambda)\hat{\gamma}_{\mu,\nu} R(\Lambda)^\dagger = \Lambda^{-1}\hat{\gamma}\tilde{\Lambda}^{-1} \quad (56)$$

Introduce the squared mass operator M^2 by

$$c^2\hbar^{-2}M^2 = \sum_{\mu,\nu} \hat{\gamma}_{\mu,\nu}^{-1} K_\mu K_\nu \quad (57)$$

Then one has obviously

$$R(\Lambda)M^2 R(\Lambda)^\dagger = M^2 \quad (58)$$

Note that M^2 is not necessarily invariant under shifts in spacetime.

Similarly, the squared eigentime operator

$$\sum_{\mu,\nu} \hat{\gamma}_{\mu,\nu}^{-1} Q_\mu Q_\nu \quad (59)$$

is also invariant under proper Lorentz transformations.

X. GAUGE TRANSFORMATIONS

Consider the gauge transformation

$$\begin{aligned} A_\mu &\rightarrow A'_\mu = A_\mu + \sum_\nu \gamma_{\mu,\nu} \frac{\partial \chi}{\partial q_\nu} \\ \Omega_\mu &\rightarrow \Omega'_\mu = \Omega_\mu - (e \cdot m) \frac{2b^2}{\hbar l^2} \sum_\nu \gamma_{\mu,\nu} \frac{\partial \chi}{\partial k_\nu} \end{aligned} \quad (60)$$

with χ an arbitrary function of k, q, e, m . Under this transformation E_α and B_α , as given by (33, 34), are invariant. Also the commutation relations (42, 43, 44) are invariant. Now let

$$U'(k, q) = \exp(-ik \cdot \hat{\gamma}^{-1} Q' + i\hat{\gamma}^{-1} q \cdot K') \quad (61)$$

with Q'_μ and K'_μ derived from (29) by substituting A_μ by A'_μ . Then U' is again a projective representation of the covariance system. It involves the same operator valued phase factor $\xi(k, q; k', q')$ because the latter depends only on the commutation relations (42, 43, 44).

Fix a positive number κ , which is the rest mass of the particle in units \hbar . Assume that $\psi(k, q, e, m)$ is a solution of the eigenvalue problem

$$\hbar^{-2}M^2\psi = \kappa^2\psi \quad (62)$$

(we do not assume that ψ is a wavefunction belonging to the Hilbert space with scalar product (24)). Then the function $\psi'(k, q, e, m)$ given by

$$\begin{aligned} & \psi'(k, q, e, m) \\ &= \exp\left(i\frac{b}{\hbar c}(e \cdot m)\chi(k, q, e, m)\right)\psi(k, q, e, m) \end{aligned} \quad (63)$$

is a solution of the eigenvalue problem

$$\hbar^{-2}(M')^2\psi' = \kappa^2\psi' \quad (64)$$

This property is what one understands by gauge invariance of the model. In order to check that (64) holds let us calculate

$$\begin{aligned} \hbar^2\kappa^2\psi' &= \exp\left(i\frac{b}{\hbar c}\sigma_3\hat{\chi}\right)\hbar^2\kappa^2\psi \\ &= \exp\left(i\frac{b}{\hbar c}\sigma_3\hat{\chi}\right)M^2\psi \\ &= \exp\left(i\frac{b}{\hbar c}\sigma_3\hat{\chi}\right)M^2\exp\left(-i\frac{b}{\hbar c}\sigma_3\hat{\chi}\right)\psi \end{aligned} \quad (65)$$

Now use that

$$K_\nu \exp\left(-i\frac{b}{\hbar c}\sigma_3\hat{\chi}\right) = \exp\left(-i\frac{b}{\hbar c}\sigma_3\hat{\chi}\right)K'_\nu \quad (66)$$

to obtain

$$M^2 \exp\left(-i\frac{b}{\hbar c}\sigma_3\hat{\chi}\right) = \exp\left(-i\frac{b}{\hbar c}\sigma_3\hat{\chi}\right)M'^2 \quad (67)$$

so that (64) follows.

XI. DISCUSSION

We have shown in this paper that the variables e and m appearing in the DFR-model can be explained as constant external fields. We have swapped the role of e and

m so that e is an electric and m is a magnetic field vector. As a side effect we have also shown that the non-vanishing time-position commutators of the model arise by substituting the spacetime position operators Q_μ by $Q_\mu - (al^2/\hbar c)\Omega_\mu$, together with the well-known substitution of momentum operators P_μ by $P_\mu - (b/c)A_\mu$. The vector potentials Ω_μ and A_μ are strictly linked because the field tensors $\frac{\partial\Omega_\mu}{\partial k_\nu}$ and $\frac{\partial A_\mu}{\partial q_\nu}$ are each others inverses, up to a constant factor. Obviously, these findings are of interest in a more general context than that of this particular model. In the present model the k - and q -dependence of Ω_μ resp. A_μ is trivial. In a more general context, we expect more complex dependency on k and q . In particular, nontrivial spacetime dependence of A_μ will lead to spacetime dependence of Ω_μ .

Projective representations with operator valued phase factors play an important role in the present paper. A more systematic study of this kind of representations is required. Also other aspects of the model require further investigation. In particular, we can make the following remarks.

- Throughout the paper the metric tensor g has been replaced by an operator \hat{g} because the mathematics allows to do so. It is not clear what such an operator-valued metric tensor means.
- We did not consider spin of the particle. Introduction of a Dirac-like equation will be discussed in a subsequent paper.
- We did not consider the problem of reducibility of the covariant representation.

-
- [1] M. Born, *Reciprocity theory of elementary particles*, Rev. Mod. Phys. **21**, 463-473 (1949).
- [2] J. Schwinger, *The Theory of Quantized Fields. I*, Phys. Rev. **82**, 914-927 (1951).
- [3] J. Madore, *Fuzzy Physics*, Ann Phys. **219**, 187-198 (1992).
- [4] S. Doplicher, K. Fredenhagen, J.E. Roberts, *Spacetime quantization induced by classical gravity*, Phys. Lett. **B331**, 39-44 (1994).
- [5] A. Kempf, *Uncertainty relation in quantum mechanics with quantum group symmetry*, J. Math. Phys. **35**, 4483-4496 (1994).
- [6] S. Doplicher, K. Fredenhagen, J.E. Roberts, *The Quantum Structure of Spacetime at the Planck Scale and Quantum Fields*, Commun. Math. Phys. **172**, 187-220 (1995).
- [7] Stephen G. Low, *Canonically relativistic quantum mechanics: representations of the unitary semidirect Heisenberg group, $U(1, 3) \otimes_s H(1, 3)$* , J. Math. Phys. **38**(5), 2197-2209 (1997).
- [8] J. Madore and J. Mourad, *Quantum space-time and classical gravity*, J. Math. Phys. **39**, 423-442 (1998).
- [9] J. Naudts, *C^* -Multipliers, crossed product algebras, and canonical commutation relations*, math-ph/9907008.
- [10] J. Naudts, M. Kuna, *Covariance systems*, math-ph/0009031.